Exercise 1

a) $e := u - u_N$ Since $V_N \subseteq V$ we discretise e and get

$$(\partial_t u, v)_H + a(u(t), v) - (\partial_t u_N, v_N)_H - a(u_N(t), v_N) = 0$$

$$\Leftrightarrow \int_G \partial_t u(t) v(x)\,dx + \int_G u'(x) v'(x)\,dx - \int_G \partial_t u_N(x) \cdot v_N(x)\,dx - \int_G u_N'(x) v_N'(t)\,dx$$

write the function $v(x)$ in the discrete basis as $v(x) = \sum_{i=1}^{N} b_i \cdot v_N^i$

$$\Leftrightarrow \sum_{i=1}^{N} \int_G \partial_t u(t) (v_N)^i b_i(x)\,dx + \int_G u'(x)(v_N')^i b_i(x)\,dx - \int_G \partial_t u_N(x) \cdot (v_N)^i b_i(x)\,dx - \int_G u_N'(x) (v_N')^i b_i(x)\,dx = 0$$

$$\Leftrightarrow \sum_{i=1}^{N} \left(\int_G \left(\partial_t u(x) - \partial_t u_N(t)\right) (v_N)^i b_i(x)\,dx + \int_G \left(u'(x) - u_N'(x)\right) (v_N')^i b_i(x)\,dx \right) = 0$$

$$\Leftrightarrow \int_G \partial_t u(x) - \partial_t u_N(x)\, v_N(x)\,dx + \int_G (u'(x) - u_N'(x)) v_N'\,dx = 0$$

$$\Leftrightarrow (\partial_t e(t), v_N(t))_H + a(e(x), v_N(x)) = 0 \qquad \square$$

b)

(i) $$(\partial_t \theta(t), w_N)_H + a(\theta(t), w_N) = \int_G \partial_t (u_N - v_N)(x) \cdot w_N(x)\,dx + \int_G (u_N - v_N)'(x)\, w_N'(x)\,dx$$

$$= \sum_{i=1}^{N} \int_G \partial_t \left(((u_N)^i - (v_N)^i) b_i(t) \right) w_N(t)\,dt + \int ((u_N')^i - (v_N')^i) b_i'(t)\, w_N'(t)\,dt$$

$$= \sum_{i=1}^{N} \int_G \partial_t \left(((u_N)^i - (v_N)^i) b_i(t) \right) w_N(x)\,dt + \int_G ((u_N')^i - (v_N')^i) b_i'(t)\, w_N'(t)\,dt -$$

$$\overbrace{(\partial_t e(x), w_N(x))_H + a(e(x), v_N(x))}^{=0}$$

$$= \sum_{i=1}^{N} \int_G \partial_t \left(((u_N)^i - (v_N)^i - (u_N)^i) b_i(t) + u(t) \right) w_N(x)\,dx + \int_G ((u_N')^i - (v_N')^i - (u_N')^i) b_i'(x) + u'(x))\, w_N'\,dx$$

$$= \int_G \partial_t (u(t) - v_N(t))\, w_N(t)\,dt + \int_G (u'(t) - v_N'(t))\, w_N'(t)\,dt$$

$$= (\partial_t \rho(t), w_N)_H + a(\rho(t), w_N) \quad \text{a.e. } t \in [0, T]$$

(ii) We now choose $w_N = \theta(t)$

Equation in (i) becomes: $\underline{(\partial_t \theta(t), \theta(t))_H} + \|\theta(t)\|_a^2 = \underline{(\partial_t \rho(t), \theta(t))}_H + a\underline{(\rho(t), \theta(t))}$

Applying Young to the —— : $(\partial_t \rho(t), \theta(t))_H \leq \frac{1}{2\varepsilon}\|\partial_t \rho(t)\|_H^2 + \frac{\varepsilon}{2}\|\theta(t)\|_H^2$ for some $\varepsilon > 0$

—— : $a(\rho(t), \theta(t)) \leq \frac{1}{2\varepsilon}\|\rho(t)\|_a^2 + \frac{\varepsilon}{2}\|\theta(t)\|_a^2$ for some $\varepsilon > 0$

LHS: $(\partial_t \theta(t), \theta(t))_H = \int_0^T \partial_t \theta(t)\, \theta(t)\, dt = |\theta(t)|^2 \Big|_0^T - \int_0^T \partial_t \theta(t)\, \theta(t)\, dt$

$\Rightarrow (\partial_t \theta(t), \theta(t))_H = \frac{1}{2}|\theta(T)|^2 - \frac{1}{2}|\theta(0)|^2 \underset{\text{Hauptsatz}}{=} \frac{1}{2}\frac{d}{dt}\int_0^T |\theta(t)|^2\, dt = \frac{1}{2}\frac{d}{dt}\|\theta(t)\|_H^2$

We thus get: $\frac{1}{2}\frac{d}{dt}\|\theta(t)\|_H^2 + \|\theta(t)\|_a^2 \leq \underbrace{\frac{1}{2\varepsilon}}_{C}\left(\|\partial_t \rho(t)\|_H^2 + \|\rho(t)\|_a^2 + \varepsilon^2\|\theta(t)\|_a^2\right)$

$$\xrightarrow{\varepsilon \to 0} C\left(\|\partial_t \rho(t)\|_H^2 + \|\rho(t)\|_a^2\right)$$

(iii)

$\|\theta\|_{L^\infty((0,T);H)} = \operatorname*{ess\,sup}_{t \in (0,T)} \left(\int_\Omega |\theta(t)|^2\, dt\right)^{\frac{1}{2}} = \operatorname*{ess\,sup}_{t \in [0,T]} \|\theta(t)\|_H$

$\|\theta\|_{L^2((0,T);V)} = \int_0^T \int_\Omega |\theta'(x)|^2\, dx\, dt = \int_0^T \|\partial_t \theta(t)\|_H^2\, dt = \int_0^T \|\theta(t)\|_a^2\, dt$

We want to apply Gronwall's lemma: $y := \|\theta\|_H^2$

From (ii) we can easily show that

$$y'(t) = \frac{d}{dt}\|\theta(t)\|_H^2 \leq 2C\left(\|\partial_t \rho(t)\|_H^2 + \|\rho(t)\|_a^2\right) - \|\theta(t)\|_a^2$$

non neg. of a norm

$$\leq \underbrace{2C\left(\|\partial_t \rho(t)\|_H^2 + \|\rho(t)\|_a^2\right) - \|\theta(t)\|_a^2}_{:= B(t)} + \|\theta(t)\|_H^2$$

$$= B(t) + y(t)$$

Thus the Gronwall lemma applies and we get:

$$y(t) = \|\theta(t)\|_H^2 \leq e^{t-0}\|\theta(0)\|_H^2 + \int_0^t e^{t-s}\left(2C\left(\|\partial_s \rho(s)\|_H^2 + \|\rho(s)\|_a^2\right) - \|\theta(s)\|_a^2\right) ds$$

(monot. of exp. + non neg of $\|\cdot\|_a$ ⇒ integral monotonicity)

$$\leq e^T\left(\|\theta(0)\|_H^2 + \int_0^T 2C\|\partial_s \rho(s)\|_H^2\, ds + \int_0^T \|\rho(s)\|_a^2\, ds\right) - \|\theta\|_{L^2((0,T);V)}$$

$$\leq C\left(\|\theta(0)\|_H^2 + T \operatorname*{ess\,sup}_{s \in [0,T]} \|\partial_s \rho\|_H^2 + \|\rho\|_{L^2((0,T);V)}^2\right) - \|\theta\|_{L^2((0,T);V)}$$

$$= C\left(\|\theta(0)\|_H^2 + \|\partial_s \rho\|_{L^2((0,T);H)}^2 + \|\rho\|_{L^2((0,T);V)}^2 - \|\theta\|_{L^2((0,T);V)}\right) \quad \square$$

(iv) $e = \rho - \theta \quad | \; \rho = u - v_N, \; \theta = u_N - v_N, \; e = \rho - \theta = u - u_N$

$$\|u - u_N\|^2_{L^\infty((0,T);H)} + \|u - u_N\|^2_{L^2((0,T);V)} = \|\rho - \theta\|^2_{L^\infty((0,T);H)} + \|\rho - \theta\|^2_{L^2((0,T);V)}$$

$$\leq \|\rho\|^2_{L^\infty((0,T);H)} + \|\rho\|^2_{L^2((0,T);V)} + \underbrace{\|\theta\|^2_{L^\infty((0,T);H)}}_{(iii)} + \|\theta\|^2_{L^2((0,T);V)}$$

$$\leq \|\rho\|^2_{L^\infty((0,T);H)} + \|\rho\|^2_{L^2((0,T);V} + C\left(\|\theta(0)\|^2_H + \|\partial_t \rho\|^2_{L^2((0,T);H)} + \|\rho\|^2_{L^2((0,T);V}\right)$$

$$\leq C\left(\|\theta(0)\|^2_H + \|\partial_t \rho\|^2_{L^2((0,T);H)} + \|\rho\|^2_{L^2((0,T);V)} + \|\rho\|^2_{L^\infty((0,T);H)}\right)$$

$$\leq C\Big(\|u_N(0) - v_N(0)\|^2_H + \|\partial_t(u - v_N)\|^2_{L^2((0,T);H)} + \|u - v_N\|^2_{L^2((0,T);V)} + \|u - v_N\|^2_{L^\infty((0,T);H)}\Big).$$

$\square$

(v) Because then the Galerkin approximation error is controlled by the best possible test-function error u-v_N and that means that there a.s. exists no better approximation

2. Finite element method for heat equation:

$$(\mathrm{I}) \begin{cases} \partial_t u(t,x) - \partial_x(\kappa(x)\partial_x u(t,x)) &= f(t,x) \quad \text{in } J \times G, \\ u(t,x) &= 0 \quad \text{on } J \times \partial G, \\ u(0,x) &= u_0(x) \quad \text{in } G. \end{cases}$$

a) Assume $u \in C^\infty((J \times G), \mathbb{R})$

Multiply with test function: $\partial_t u(t,x) \cdot v(t,x) - \partial_x(\kappa(x)\partial_x u(t,x)) \cdot v(t,x) = f(t,x) \cdot v(t,x)$
$v \in L^2(G)$

Integrate over G: $\int_G \partial_t u(t,x) v(x)\, dx - \int_G \partial_x(\kappa(x)\partial_x u(t,x) \cdot v(x))\, dx = \int_G f(t,x) \cdot v(x)\, dx$

Apply partial integration to ___ :

$$\frac{d}{dt}(u(t), v)_{L^2(G)} + \int_G \kappa(x)\, \partial_x u(t,x)\, \partial_x v(t,x)\, dx - \underbrace{\int_{\partial G} \kappa(x)\, \partial_x u(t,x)\, v(x)\, d\sigma(x)}_{=0} = (f(t), v)_{L^2(G)}$$

(*) $\frac{d}{dt}(u(t), v)_{L^2(G)} + a(u(t), v) = (f(t), v)_{L^2(G)}$

where $a(\varphi, \psi) := \int_G \kappa(x)\partial_x\varphi(x)\partial_x\psi(x)\, dx$

b) $\varkappa(x) \geq 0 \ \forall x \in \bar{G}$ $\quad V_N \subset H_0^1(G)$

N. We seek an approximation u_N of the solution of system (1) as the element of $C^1([0,T];V_N)$ satisfying the variational problem

$$\frac{d}{dt}(u_N(t), v_N)_{L^2(G)} + a(u_N(t), v_N) = (f(t), v_N)_{L^2(G)}\,, \quad \forall v_N \in V_N\,,\ \forall t \in J\,, \qquad (2)$$

with initial condition $u_N(0,x) = u_{0,N}$ where $u_{0,N}$ is some approximation of u_0 in V_N (this is sometimes called the "method of lines"). Given a basis $\{\phi_{N,j}\}_{1\leq j\leq N}$ of V_N, write

$$u_N(t,\cdot) = \sum_{j=1}^{N} u_{N,j}(t)\phi_{N,j}\,, \quad \underline{u}_N(t) = \begin{pmatrix} u_{N,1}(t) \\ \vdots \\ u_{N,N}(t) \end{pmatrix}.$$

Show that the vector $\underline{u}_N(t)$ satisfies a system of coupled ordinary differential equations of the form:

$$\mathbf{M}\frac{d}{dt}\underline{u}_N + \mathbf{A}\underline{u}_N = \underline{F}\,. \qquad (3)$$

Give the expression of the matrices $\mathbf{M}$, $\mathbf{A}$ and the vector $\underline{F}$. Why are $\mathbf{M}$, $\mathbf{A}$ nonsingular?

Plugging into (2) the definition $u_N(t,\cdot)$:

$$\frac{d}{dt}\int_G u_N(t)\cdot v_N\,dx + \int_G \partial_x u_N(t)\,\partial_x v_N\,dx = \int_G f(t)\,v_N\,dx$$

(Einstein sum notation)

$$\Leftrightarrow \frac{d}{dt}\int_G u_{N,j}(t)\phi_{N,j}\,v_{N,i}\,\phi_{N,i}\,dx + \int_G \varkappa(x)\partial_x\phi_{N,j}\,u_{N,j}\,\partial_x\phi_{N,i}\,v_{N,i}\,dx = \int_G f(t)\,\phi_{N,i}\,v_{N,i}\,dx$$

define $\underline{M} = \left[\int_G \phi_{N,i}\,\phi_{N,j}\,dx\right]_{i,j=1}^{N,N}$ $\quad \underline{A} = \left[\int_G \varkappa(x)\partial_x\phi_{N,i}\,\partial_x\phi_{N,j}\,dx\right]_{i,j=1}^{N}$ $\quad \underline{F} = \left[\int_G \phi_{N,i}\right]_{i=1}^{N}$

$$\Leftrightarrow \underline{v}_N^T\,\underline{M}\cdot\frac{d}{dt}\underline{u}_N + \underline{v}_N^T\cdot\underline{A}\cdot\underline{u}_N = \underline{v}_N^T\cdot\underline{F}$$

$$\Leftrightarrow \underline{M}\frac{d}{dt}\underline{u}_N + \underline{A}\,\underline{u}_N = \underline{F} \qquad \square$$

c) We seek approximations $u_{N,i}^m$ of the values of the coefficients $u_{N,i}(t_m)$ at each time $t_m = km$ where $k > 0$ is the time step and $m \in \mathbb{N}$. Let

$$\underline{u}_N^m := \begin{pmatrix} u_{N,1}^m \\ \vdots \\ u_{N,N}^m \end{pmatrix}.$$

Starting from Eq. (3), proceed as in the case of the finite difference scheme and derive a fully discrete scheme for (2) of the form

$$\mathbf{B}_\theta\underline{u}_N^{m+1} = \mathbf{C}_\theta\underline{u}_N^m + \underline{F}_\theta^m\,,$$

with suitable matrices $\mathbf{B}_\theta$, $\mathbf{C}_\theta$ and $\underline{F}_\theta^m$.

$$(3)\ \underline{M}\frac{d}{dt}\underline{u}_N^m + \underline{A}\,\underline{u}_N^m = \underline{F}^m \approx \underline{M}\cdot\frac{\underline{u}_N^{m+1}\theta - \underline{u}_N^m(1-\theta)}{k} + \underline{A}\,\underline{u}_N^m = \underline{F}^m$$

$$\Leftrightarrow \underbrace{\theta\underline{M}}_{:=\underline{B}_\theta}\underline{u}_N^{m+1} = \underline{F}\cdot k - \underline{A}\cdot\underline{u}_N^m\cdot k + \underline{M}\,\underline{u}_N^m(1-\theta)$$

$$\underline{B}_\theta = \underbrace{((1-\theta)\underline{M} - k\cdot\underline{A})}_{:=\underline{C}_\theta}\underline{u}_N^m + \underbrace{\underline{F}^m k}_{:=\underline{F}_\theta^m} \qquad \square$$

3 Implementation in Python:

$J = G = (0,1) \qquad \kappa(x) = x+1$

From now on, we assume $J = G = (0,1)$ and $\kappa(x) := x+1$. For any $N, M \in \mathbb{N}$, we set $h = \frac{1}{N+1}$, $k = \frac{1}{M}$ and consider the spatial mesh points $x_i = hi, i = 1,2,\dots,N$. Let V_N be the vector space of continuous functions on G, vanishing at both ends of the interval, and which are linear on each (x_i, x_{i+1}). For each $i \in \{1,\dots,N\}$, there is a unique element $\phi_{N,i}$ of V_N (the so-called hat-functions) satisfying

$$\phi_{N,i}(x_j) = \delta_{i,j}\,, \quad \forall j \in \{1,\dots,N\}$$

and $\{\phi_{N,i}\}_{1\le i\le N}$ is a basis of V_N.

a) Let $K(x,y) := \int_x^y \kappa(x)\,dx$. Derive an explicit expression for K in terms of $x, y \in \mathbb{R}$. In the script "`FEM_heat.py`" provided, implement the function "`kappa_integral(x,y)`" for the calculation of $K(x,y)$ using the derived explicit expression.

$$K(x,y) = \int_x^y \kappa(x)\,dx = \int_x^y x+1\,dx = \left[\tfrac{1}{2}x^2 + x\right]_x^y = \tfrac{1}{2}(y^2 - x^2) + (y-x)$$

b)

$$\underline{M} = \left[\int_G \phi_{N,i}\,\phi_{N,j}\,dx\right]_{i,j=1}^{N,N} \qquad \underline{A} = \left[\int_G \kappa(x)\,\partial_x\phi_{N,i}\,\partial_x\phi_{N,j}\,dx\right]_{i,j=1}^{N}$$

with the hat basis functions we have $\phi_{N,i}\cdot\phi_{N,j} = \delta_{ij}$ $\quad i,j \in \{0,\dots,N\}$

so $\underline{M} = \underset{1}{|G|}\,\underline{\underline{I}} = \underline{\underline{I}}$

$$\partial_x \phi_{Ni} = \begin{cases} \frac{1}{h} & \text{if } x \in ((h-1)i, hi) \\ -\frac{1}{h} & \text{if } x \in (hi, (h+1)i) \\ 0 & \text{if } x \in \mathbb{R} \setminus [(h-1)i, (h+1)i] \end{cases}$$

$$\partial_x\phi_{Ni}\cdot\partial_x\phi_{Nj} = \begin{cases} \partial_x\phi_{Ni} & \text{if } i=j \\ -\mathbb{1}_{x\in(hi,h(i+1))} & \text{if } i-j=1 \text{ a.e} \\ -\mathbb{1}_{x\in(h(i-1),hi)} & \text{if } i-j=-1 \text{ a.e} \\ 0 & \text{else a.e} \end{cases}$$

γ)

c) Let $u(t,x) = e^{-t}\sin(\pi x)$. Check that $u(t,x)$ is the solution of system (1) for

$$f(t,x) = \left((x+1)\pi^2 - 1\right)e^{-t}\sin(\pi x) - \pi e^{-t}\cos(\pi x)\,, \quad u_0(x) = \sin(\pi x)\,.$$

Define the corresponding functions "`f(t,x)`", "`initial_value(x)`" and "`exact_solution_at_1(x)`" in "`FEM_heat.py`". Here "f" has the temporal and spatial variables (t,x) as the input and outputs the value of $f(t,x)$. "initial_value" and "exact_solution_at_1" shall receive a vector of spatial grid points and compute a vector containing the value of $u(x,0)$ and $u(x,1)$ at these points respectively.

d) Show that

$$\int_G f(t,x)\phi_{N,i}(x)\,dx = h\frac{f(t,x_i - h/2) + f(t,x_i) + f(t,x_i + h/2)}{3} + O(h^5)\,. \qquad (4)$$

In the template "`FEM_heat.py`", complete the function "`build_F(t,N)`" accordingly. The parameters of this function are the time level t and the discretization parameter N. The output shall be the approximated value of the column vector $\underline{F}$ at time t, using (4).